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Classification of Type II code over GF(4) of some small lengths

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1 Introduction

Let $\mathrm{GF}(2^r)$ be the finite field with 2^r elements. Let C be a code over $\mathrm{GF}(2^r)$ of length n, which is a subspace of the vector space $\mathrm{GF}(2^r)^n$. Let $B=\{b_1,b_2,\ldots,b_r\}$ be a basis of $\mathrm{GF}(2^r)$ over $\mathrm{GF}(2)$. We denote by $\phi_B(a)=(a_1,a_2,\ldots,a_r)\in\mathrm{GF}(2)^r$ the representation of $a\in\mathrm{GF}(2^r)$ over $\mathrm{GF}(2)$ with respect to a basis B, that is, $a=\sum_{i=1}^r a_ib_i$. For $u=(u_1,u_2,\ldots,u_n)\in\mathrm{GF}(2^r)^n$, we also denote by $\phi_B(u)$ the vector in $GF(2)^{rn}$ obtained by concatenating $\phi_B(u_1),\ldots,\phi_B(u_n)$. We call $\phi_B(C)$ the *binary image* of C with respect to B. B is called a *trace-orthogonal basis* (TOB) if $\mathrm{Tr}(b_ib_j)=\delta_{ij}$ for $1\le i,j\le r$ where Tr denotes the trace function of $\mathrm{GF}(2^r)$ over $\mathrm{GF}(2)$.

In 1980's, Pasquier and Wolfmann studied self-dual codes over $\mathrm{GF}(2^r)$ whose binary images with respect to a TOB are binary Type II codes (that is, binary doubly-even self-dual codes) including the extended Hamming code and the extended Golay code (cf. [8, 10]). We say that such codes over $\mathrm{GF}(2^r)$ are Type II codes with respect to a TOB [4] (see [6] for Type II codes over GF(4)). Recently, it has been proved by the author that the Type II property with respect to a TOB for self-dual codes over $\mathrm{GF}(2^r)$ is independent of the choice of a TOB [1]. This allows us to call C a Type II code if C is a Type II code with respect to a TOB, without a reference to an explicit TOB. The *binary length* of a code over $\mathrm{GF}(2^r)$ of length n is defined by rn, which is the length of its binary image. The Type II codes with binary length up to 24 have been classified (cf. [2, 4, 6]). We refer to [9] for the classification of binary Type II codes. The next problem would be to classify all Type II codes over $\mathrm{GF}(2^r)$ for any r with binary length 32.

Theorem 1.1 (Munemasa [7]) *The total number of Type II* $\mathrm{GF}(2^r)$*-codes of length* n *is given by*

$$N_{II,r}(n)=\prod_{i=0}^{n/2-2}(2^{ri}+1),\tag{1}$$

if $rn\equiv 0 \pmod 8$ *and* $n\equiv 0 \pmod 4$, *and* 0 *otherwise.*

The formula (1) is called the mass formula for Type II codes over GF(2^r). By Theorem 1.1, the possible cases for which there is a Type II code over GF(2^r) of length n with binary length 32 are $(n,r) = (32,1)$, $(16,2)$, $(8,4)$ and $(4,8)$. For the cases $(n,r) = (32,1)$ and $(n,r) = (8,4)$, the complete classifications are given in [5] and [4], respectively. Furthermore, for the case $(n,r) = (4,8)$, it is shown that there exists a unique Type II code up to permutation-equivalence [7].

In this paper, we give a classification for the case $(n,r) = (16,2)$, that is, Type II codes over GF(4) of length 16. It is the only remaining case to complete the classification of Type II codes with binary length 32.

2 Classification of Type II Codes over GF(4) of Length 16

In this section, we give a classification of Type II codes of length 16 over $\mathrm{GF}(4) = \mathrm{GF}(2)[\omega]/(\omega^2+\omega+1)$. A *bisorted* matrix is constructed by sorted vectors on a lexicographical order with both directions North to South and West to East. We check all the bisorted 8×8 GF(4)-matrices A's such that $(\ I\ A\ )$ generates a Type II code, where I is the 8×8 identity matrix. It is sufficient to consider such matrices to complete the classification [3]. Indeed, we obtain 82588 distinct Type II codes by the method above. Let C be a code over GF(4) of length 16, and let

$$\widetilde{C} = \{(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{16}) \mid \mathrm{wt}(c) = 6, c = (c_1, c_2, \ldots, c_{16}) \in C\},$$

where $\mathrm{wt}(c)$ is the Hamming weight of c and $\hat{c}_i = 0$ if $c_i = 0$ and $\hat{c}_i = 1$ otherwise. Then $\widetilde{C}$ is a non-linear binary code of length 16. We now consider three invariants of codes under the permutation-equivalence:

1. the Hamming weight enumerator $W(C)$,
2. the order $|\operatorname{Aut}(C)|$ of the permutation automorphism group of C,
3. the order $|\operatorname{Aut}(\widetilde{C})|$ of the permutation automorphism group of $\widetilde{C}$.

Calculating the invariants for every code using MAGMA, we find 48 codes $D_1, \ldots, D_{48}$ with distinct sets of invariants. The minimum Hamming weight $d(D_i)$ of D_i, the Hamming weight enumerator $W(D_i)$ of D_i and the values $|\operatorname{Aut}(D_i)|$ are listed in Table 1. The weight enumerators are listed in Table 2 where only the coefficients of the monomials $x^{16-i}y^i$ for $i = 3, 4, 6, 7, \ldots, 16$ are given. For all weight enumerators, the coefficients of the monomials x^{16}, $x^{15}y$, $x^{14}y^2$ and $x^{11}y^5$ are 1, 0, 0 and 0, respectively. We verified that

Table 1: Properties of the Type II codes over GF(4) of length 16

	$d(D_i)$	$W(D_i)$	$\lvert\mathrm{Aut}(D_i)\rvert$		$d(D_i)$	$W(D_i)$	$\lvert\mathrm{Aut}(D_i)\rvert$
D_1	3	W_1	497664	D_{25}	4	W_{24}	5160960
D_2	3	W_2	387072	D_{26}	4	W_{25}	2304
D_3	3	W_3	11664	D_{27}	4	W_{26}	73728
D_4	3	W_4	5184	D_{28}	4	W_{27}	6144
D_5	3	W_5	648	D_{29}	4	W_{28}	576
D_6	3	W_6	7776	D_{30}	4	W_{28}	6144
D_7	3	W_7	324	D_{31}	4	W_{28}	3072
D_8	3	W_8	13608	D_{32}	4	W_{29}	18432
D_9	3	W_9	576	D_{33}	4	W_{29}	18432
D_{10}	3	W_{10}	90	D_{34}	4	W_{30}	128
D_{11}	3	W_{11}	216	D_{35}	4	W_{30}	64
D_{12}	3	W_{12}	432	D_{36}	4	W_{31}	240
D_{13}	3	W_{13}	3456	D_{37}	4	W_{32}	24
D_{14}	3	W_{14}	34560	D_{38}	4	W_{32}	144
D_{15}	3	W_{15}	276480	D_{39}	4	W_{32}	48
D_{16}	3	W_{16}	13824	D_{40}	4	W_{33}	48
D_{17}	3	W_{17}	12096	D_{41}	4	W_{33}	96
D_{18}	3	W_{18}	288	D_{42}	6	W_{34}	96
D_{19}	3	W_{19}	216	D_{43}	6	W_{34}	384
D_{20}	3	W_{20}	36	D_{44}	6	W_{35}	16
D_{21}	3	W_{21}	192	D_{45}	6	W_{35}	336
D_{22}	3	W_{22}	7920	D_{46}	6	W_{35}	8
D_{23}	3	W_{23}	18	D_{47}	6	W_{35}	14
D_{24}	4	W_{24}	3612672	D_{48}	6	W_{35}	24

Table 2: The weight enumerators

i	3	4	6	7	8	9	10	11	12	13	14	15	16
W_1	48	12	864	432	54	6912	5184	1296	20844	20736	7776	1296	81
W_2	24	48	312	1080	306	4032	7056	4104	24840	12096	7992	3240	405
W_3	24	3	522	360	351	5952	6156	5544	18165	17856	7722	2520	360
W_4	18	12	384	522	414	5232	6624	6246	19164	15696	7776	3006	441
W_5	12	0	360	300	534	5376	6816	7428	17136	16128	7896	3012	537
W_6	12	3	378	252	603	5184	7164	6948	17757	15552	8298	2772	612
W_7	9	3	282	405	531	5112	6876	8019	17325	15336	7722	3375	540
W_8	9	21	150	837	405	4440	6804	8019	19731	13320	7254	4095	450
W_9	6	6	252	366	636	4752	7368	8034	17778	14256	8124	3306	651
W_{10}	6	0	264	318	606	5040	7104	8418	16800	15120	7896	3354	609
W_{11}	6	3	234	414	567	4944	7020	8514	17157	14832	7722	3546	576
W_{12}	6	9	222	462	597	4656	7284	8130	18135	13968	7950	3498	618
W_{13}	6	18	132	750	480	4368	7032	8418	19206	13104	7428	4074	519
W_{14}	6	30	204	558	756	3600	8424	6498	21690	10800	9036	3114	819
W_{15}	12	48	120	1116	450	3360	7632	6084	24168	10080	7992	3924	549
W_{16}	12	12	288	540	486	4896	6912	7236	18828	14688	7776	3348	513
W_{17}	3	30	156	567	792	3432	8568	6993	21522	10296	9036	3285	855
W_{18}	3	12	144	567	594	4392	7344	8721	18324	13176	7776	3861	621
W_{19}	3	3	234	279	711	4680	7596	8433	17253	14040	8298	3285	720
W_{20}	3	3	186	423	603	4776	7164	9009	16989	14328	7722	3717	612
W_{21}	3	6	204	375	672	4584	7512	8529	17610	13752	8124	3477	687
W_{22}	12	3	330	396	495	5280	6732	7524	17493	15840	7722	3204	504
W_{23}	3	0	216	327	642	4872	7248	8913	16632	14616	7896	3525	645
W_{24}	0	84	336	0	1854	0	14112	0	32004	0	15120	0	2025
W_{25}	0	18	228	192	984	3648	9048	7104	19926	10944	9732	2688	1023
W_{26}	0	36	48	768	750	3072	8544	7680	22068	9216	8688	3840	825
W_{27}	0	24	168	384	906	3456	8880	7296	20640	10368	9384	3072	957
W_{28}	0	12	96	576	630	4224	7488	9216	18156	12672	7776	4032	657
W_{29}	0	12	288	0	1062	3840	9216	6912	19212	11520	10080	2304	1089
W_{30}	0	6	156	384	708	4416	7656	9024	17442	13248	8124	3648	723
W_{31}	0	9	126	480	669	4320	7572	9120	17799	12960	7950	3840	690
W_{32}	0	3	138	432	639	4608	7308	9504	16821	13824	7722	3888	648
W_{33}	0	3	186	288	747	4512	7740	8928	17085	13536	8298	3456	756
W_{34}	0	0	216	192	786	4608	7824	8832	16728	13824	8472	3264	789
W_{35}	0	0	168	336	678	4704	7392	9408	16464	14112	7896	3696	681

$|\operatorname{Aut}(\widetilde{D}_{32})|$ and $|\operatorname{Aut}(\widetilde{D}_{33})|$ are 7962624 and 36864, respectively. By Table 1, $D_1, \dots, D_{48}$ are not permutation-equivalent. By Theorem 1.1, we have that

$$\begin{aligned}\sum_{i=1}^{48} \frac{16!}{|\operatorname{Aut}(D_i)|} &= 11925737086250 \\ &= N_{II,2}(16).\end{aligned}$$

Thus, we obtain the following:

Theorem 2.1 *There exist* 48 *Type II codes over* GF(4) *of length* 16*, up to permutation-equivalence.*

Generator matrices of the codes are given in Section 3. The Frobenius automorphism is the field automorphism on GF(4) defined by $a \mapsto a^2$. Each of these 48 codes is permutation-equivalent to its Frobenius image. Finally, we calculate the binary images of the codes using MAGMA. In Table 3, the binary codes $\phi(C)$ are given and the minimum Hamming weights $d(\phi(C))$ of $\phi(C)$ are also given. We use the notation in [9] for the binary Type II codes of length 32. There exist 7 Type II codes whose binary images are of minimum Hamming weight 8 up to permutation-equivalence. Only 4 codes among the 5 extremal binary Type II codes of length 32 are the binary images of Type II codes over GF(4).

As a consequence, we obtain the complete classification of Type II codes over GF(2^r) with binary length 32. The numbers of Type II codes over GF(2^r) with binary length 32 up to permutation-equivalence are in Table 4. In the table, #1, #2 and #3 denote the number of codes up to permutation-equivalence, the number of codes whose binary images are extremal, and the number of extremal binary codes obtained as the binary images, respectively.

3 Generator Matrices

In order to save space, we list generator matrices ($I\ \{a_{i,j}\}$) as

$$a_{1,1}a_{1,2}\cdots a_{1,8}, a_{2,1}\cdots a_{2,8}, \dots, a_{8,1}\cdots a_{8,8},$$

where $\bar{\omega}$ denotes ω^2.

D_1: $000000\omega\bar\omega$, $000000\bar\omega\omega$, $0000\omega\bar\omega00$, $0000\bar\omega\omega00$, $00\omega\bar\omega0000$, $00\bar\omega\omega0000$, $\omega\bar\omega000000$, $\bar\omega\omega000000$

D_2: $000000\omega\bar\omega$, $000000\bar\omega\omega$, 00011100, 00101100, 00110100, 00111000, $\omega\bar\omega000000$, $\bar\omega\omega000000$

D_3: $000000\omega\bar\omega$, $000000\bar\omega\omega$, $0000\omega\bar\omega00$, $00\omega\bar\omega0000$, $0\omega\bar\omega\omega1\bar\omega00$, $0\bar\omega\omega1\bar\omega\omega00$, $\omega01\bar\omega\bar\omega\omega00$, $\bar\omega0\bar\omega\omega\omega100$

D_4: $000000\omega\bar\omega$, $000000\bar\omega\omega$, 00011100, 00101100, $0\omega\bar\omega\bar\omega\omega100$, $0\bar\omega\omega\omega1\bar\omega00$, $\omega0\bar\omega\bar\omega1\omega00$, $\bar\omega0\omega\omega\bar\omega100$

D_5: $000000\omega\bar\omega$, $0000\omega\bar\omega00$, $000\omega\bar\omega\omega1\bar\omega$, $00\omega01\bar\omega\bar\omega\omega$, $0\omega\bar\omega100\bar\omega\omega$, $0\bar\omega\omega\bar\omega00\omega1$, $\omega01\bar\omega\bar\omega\omega00$, $\bar\omega0\bar\omega\omega\omega100$

Table 3: Binary images of the Type II code over GF(4) of length 16

$\phi(D_i)$	$d(\phi(D_i))$	Codes over GF(4)	$\phi(D_i)$	$d(\phi(D_i))$	Codes over GF(4)
C5	4	D_{25}	C59	4	D_{11}, D_{29}
C10	4	D_{15}	C60	4	D_{10}, D_{36}
C17	4	D_{14}	C62	4	D_{21}
C24	4	D_1, D_2, D_{24}	C64	4	D_{19}, D_{20}
C26	4	D_3, D_4	C66	4	D_{23}
C27	4	D_{16}	C67	4	D_{30}, D_{32}, D_{33}
C28	4	D_{22}	C69	4	D_{31}
C30	4	D_{27}	C75	4	D_{34}, D_{35}
C33	4	D_{13}	C77	4	D_{41}
C34	4	D_{28}	C78	4	D_{37}, D_{38}, D_{39}, D_{40}
C45	4	D_8, D_{17}	C81	8	D_{46}
C51	4	D_6	C83	8	D_{43}, D_{45}
C53	4	D_5, D_{12}, D_{26}	C84	8	D_{47}
C55	4	D_7, D_{18}	C85	8	D_{42}, D_{44}, D_{48}
C57	4	D_9			

Table 4: The number of Type II codes with binary length 32

r	#1	#2	#3	reference
1	85	5	5	[5]
2	48	7	4	Section 2
4	6	1	1	[4]
8	1	0	0	[7]

D_6: $000000\omega\bar\omega$, $0000\omega\bar\omega00$, $0001\bar\omega\omega\bar\omega\omega$, $0010\bar\omega\omega\bar\omega\omega$, $0\omega\bar\omega\bar\omega00\omega1$, $0\bar\omega\omega\omega001\bar\omega$, $\omega0\bar\omega\bar\omega\omega100$, $\bar\omega0\omega\omega1\bar\omega00$

D_7: $000000\omega\bar\omega$, $0000\omega\bar\omega00$, $000\omega\bar\omega\omega1\bar\omega$, $0\omega\omega1\omega1\omega1$, $0\omega\bar\omega100\bar\omega\omega$, $0\bar\omega\omega100\bar\omega\omega$, $\omega1111\bar\omega00$, $\bar\omega\bar\omega\bar\omega\bar\omega\bar\omega\omega00$

D_8: $000000\omega\bar\omega$, $0000\omega\bar\omega00$, $0011\omega11\bar\omega$, $0101\omega11\bar\omega$, $0110\omega11\bar\omega$, 01110000, $\omega000\bar\omega\omega1\bar\omega$, $\bar\omega000\omega1\bar\omega\omega$

D_9: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $000\omega\bar\omega\omega1\bar\omega$, $0\omega\omega0\omega\omega\bar\omega\omega$, $0\omega\bar\omega\omega001\bar\omega$, $0\bar\omega\omega\omega001\bar\omega$, $\omega11\bar\omega1100$, $\bar\omega\bar\omega\bar\omega\omega\bar\omega\bar\omega00$

D_{10}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $00\omega0\bar\omega1\bar\omega\omega$, $0\omega0\bar\omega01\bar\omega\omega$, $0\omega\bar\omega0\bar\omega0\omega1$, $0\bar\omega1101\omega1$, $\omega1\bar\omega\bar\omega\omega\omega1\bar\omega$, $\bar\omega\bar\omega\omega\omega11\bar\omega\omega$

D_{11}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $00\omega1111\bar\omega$, $0\omega1\omega\bar\omega\bar\omega00$, $0\omega1\bar\omega\omega\bar\omega00$, $0\bar\omega1\bar\omega\bar\omega11\bar\omega$, $\omega110011\bar\omega$, $\bar\omega\bar\omega\bar\omega00\bar\omega\bar\omega\omega$

D_{12}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, 00111000, $0\omega1\omega\bar\omega\bar\omega00$, $0\omega1\bar\omega\omega\bar\omega00$, $0\bar\omega0\bar\omega\bar\omega\bar\omega\bar\omega\omega$, $\omega1000\bar\omega\bar\omega\omega$, $\bar\omega\bar\omega000\omega\omega1$

D_{13}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, 00111000, 01011000, $01101\omega1\bar\omega$, $01110\omega1\bar\omega$, $\omega001111\bar\omega$, $\bar\omega00\bar\omega\bar\omega\bar\omega\bar\omega\omega$

D_{14}: $000000\omega\bar\omega$, 00011100, 00101100, 01001100, $011101\bar\omega\omega$, $011110\bar\omega\omega$, $\omega000\bar\omega\bar\omega\omega1$, $\bar\omega000\omega\omega1\bar\omega$

D_{15}: $000000\omega\bar\omega$, $000000\bar\omega\omega$, 00011100, 00101100, 01001100, 10001100, $1111\omega\bar\omega00$, $1111\bar\omega\omega00$

D_{16}: $000000\omega\bar\omega$, $000000\bar\omega\omega$, 00011100, $0\omega\bar\omega\omega\bar\omega100$, $0\bar\omega\omega\omega\bar\omega100$, $1\omega\omega\bar\omega0\bar\omega00$, $1\bar\omega\bar\omega0\omega\omega00$, $111\bar\omega\omega100$

D_{17}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $000\omega\bar\omega\omega1\bar\omega$, $000\bar\omega\omega\omega1\bar\omega$, $011\omega\omega\omega\omega1$, $101\omega\omega\omega\omega1$, $110\omega\omega\omega\omega1$, 11100000

D_{18}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $000\omega\bar\omega\omega1\bar\omega$, $00\omega1111\bar\omega$, $01\bar\omega011\omega1$, $10\bar\omega011\omega1$, $11\omega\bar\omega1100$, $111\bar\omega00\omega1$

D_{19}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $00\omega0\bar\omega1\bar\omega\omega$, $0\omega0\bar\omega01\bar\omega\omega$, $0\bar\omega\bar\omega00\omega\omega1$, $1\omega1\bar\omega1100$, $1\bar\omega\bar\omega\omega\omega000$, $11\omega1\bar\omega100$

D_{20}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $00\omega0\bar\omega1\bar\omega\omega$, $0\omega01111\bar\omega$, $0\bar\omega\bar\omega\omega1\omega00$, $1\omega1\bar\omega1100$, $1\bar\omega\bar\omega\bar\omega101\bar\omega$, $11\omega0011\bar\omega$

D_{21}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, 00111000, $0\omega\omega0\omega\omega\bar\omega\omega$, $0\bar\omega\omega\omega001\bar\omega$, $1\omega\bar\omega0\bar\omega\omega00$, $1\bar\omega0\bar\omega\bar\omega11\bar\omega$, $11\bar\omega\bar\omega0\bar\omega1\bar\omega$

D_{22}: $000000\omega\bar\omega$, $000000\bar\omega\omega$, $0\omega\omega\bar\omega\bar\omega100$, $\omega0\bar\omega\omega1\bar\omega00$, $\omega\bar\omega\bar\omega0\omega100$, $\bar\omega\omega0\bar\omega1\omega00$, $\bar\omega1\omega11100$, $1\bar\omega1\omega1100$

D_{23}: $000000\omega\bar\omega$, $000\omega\omega\bar\omega1\bar\omega$, $0\omega\omega0\omega\omega\bar\omega\omega$, $\omega0\bar\omega0\omega01\bar\omega$, $\omega\bar\omega\bar\omega1\omega1\bar\omega\omega$, $\bar\omega\omega0\bar\omega00\omega1$, $\bar\omega1\omega\bar\omega\bar\omega\omega\omega1$, $1\bar\omega1\omega\omega\bar\omega\bar\omega\omega$

D_{24}: 00000111, 00001011, 00001101, 00001110, 01110000, 10110000, 11010000, 11100000

D_{25}: 00000111, 00001011, 00010011, 00100011, 01000011, 10000011, 11111101, 11111110

D_{26}: 00000111, 00001011, $000\omega\bar\omega\bar\omega\omega1$, $00\omega0\bar\omega\bar\omega1\omega$, $01\bar\omega\bar\omega00\omega\omega$, $10\bar\omega\bar\omega00\omega\omega$, $11\omega1\omega\omega0\omega$, $111\omega\omega\omega\omega0$

D_{27}: 00000111, 00001011, 00010011, $001111\omega\bar\omega$, $010111\omega\bar\omega$, $100111\omega\bar\omega$, $111\omega\omega\omega\omega0$, $111\bar\omega\bar\omega\bar\omega0\bar\omega$

D_{28}: 00000111, 00001011, 00010011, $0\omega\bar\omega\omega\omega\omega0\omega$, $0\bar\omega\omega\omega\omega\omega0\omega$, $1\omega\omega\bar\omega\bar\omega\bar\omega1\omega$, $1\bar\omega\bar\omega000\omega\omega$, $111\bar\omega\bar\omega\bar\omega\bar\omega0$

D_{29}: 00000111, 00001011, $001111\omega\bar\omega$, $0\omega\omega\bar\omega\omega\omega0\omega$, $0\bar\omega\omega\bar\omega\bar\omega\bar\omega\bar\omega0$, $1\omega\bar\omega01101$, $1\bar\omega0\omega1110$, $11\bar\omega\omega0011$

D_{30}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $000\bar\omega\omega\omega\bar\omega1$, $0\omega\bar\omega\omega\omega\omega\omega0$, $0\bar\omega\omega\omega\omega\omega\omega0$, $1\omega\omega\omega\omega011$, $1\bar\omega\bar\omega\omega\omega1\bar\omega\omega$, $111001\omega\bar\omega$

D_{31}: 00000111, $00\omega\omega10\bar\omega\bar\omega$, $010111\omega\bar\omega$, $01\omega\omega00\bar\omega\bar\omega$, $011011\omega\bar\omega$, $1\omega\bar\omega\bar\omega\omega\bar\omega\omega1$, $1\bar\omega\bar\omega\bar\omega\bar\omega110$, $11001\omega\bar\omega1$

D_{32}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $000\bar\omega\omega\omega\bar\omega1$, $0\omega\bar\omega00\omega\bar\omega1$, $0\bar\omega\omega00\omega\bar\omega1$, $1\omega\omega\omega\omega011$, $1\bar\omega\bar\omega\bar\omega\bar\omega101$, 11111110

D_{33}: 00000111, $00\omega\omega10\bar\omega\bar\omega$, $0\omega0\omega1\bar\omega0\bar\omega$, $0\omega\omega01\bar\omega\bar\omega0$, 01110000, $10\bar\omega\bar\omega00\omega\omega$, $1\bar\omega0\bar\omega0\omega0\omega$, $1\bar\omega\bar\omega00\omega\omega0$

D_{34}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $00\omega0\bar\omega1\bar\omega\omega$, $0\omega0\bar\omega0\bar\omega\omega1$, $0\bar\omega\bar\omega010\omega\omega$, $1\omega1\bar\omega0110$, $1\bar\omega\bar\omega\omega\omega1\bar\omega\omega$, $11\omega1\omega0\omega\omega$

D_{35}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $001\omega\omega\bar\omega0\bar\omega$, $0\omega\omega\omega\bar\omega\omega0\omega$, $0\bar\omega\omega\bar\omega\bar\omega\bar\omega\bar\omega0$, $1\omega\bar\omega\omega\bar\omega\bar\omega\omega1$, $1\bar\omega00\bar\omega\omega\omega0$, $11\bar\omega\omega0101$

D_{36}: 00000111, 00001011, $0\omega\omega\bar\omega\omega\omega0\omega$, $\omega0\bar\omega\omega\omega\omega\omega0$, $\omega\bar\omega\bar\omega0\bar\omega\bar\omega\bar\omega0$, $\bar\omega\omega0\bar\omega\bar\omega\bar\omega0\bar\omega$, $\bar\omega1\omega10011$, $1\bar\omega1\omega0011$

D_{37}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $0\omega\omega01\bar\omega\bar\omega0$, $\omega0\bar\omega00\bar\omega1\omega$, $\omega\bar\omega\bar\omega1\omega\omega1\bar\omega$, $\bar\omega\omega0\bar\omega\bar\omega\bar\omega\bar\omega0$, $\bar\omega1\omega\bar\omega\omega\omega\bar\omega1$, $1\bar\omega1\omega0101$

D_{38}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $0\omega\bar\omega00\omega\bar\omega1$, $\omega0\bar\omega\omega\omega\omega\omega0$, $\omega1\bar\omega\bar\omega\omega\bar\omega1\omega$, $\bar\omega\bar\omega\omega\omega11\omega\bar\omega$, $\bar\omega\bar\omega\bar\omega\omega0\bar\omega\bar\omega0$, $1\omega\bar\omega\omega\bar\omega\bar\omega\omega1$

D_{39}: 00000111, $00\omega\omega10\bar\omega\bar\omega$, $0\omega0\omega1\bar\omega0\bar\omega$, $\omega0\bar\omega\omega\omega0\omega\omega$, $\omega1\bar\omega\omega\bar\omega\omega\bar\omega1$, $\bar\omega\bar\omega001\omega0\omega$, $\bar\omega\bar\omega\omega\bar\omega0\bar\omega0\bar\omega$, $1\omega1\omega10\omega\omega$

D_{40}: 00000111, $000\omega\bar\omega\omega\bar\omega1$, $0\omega\omega01\bar\omega\bar\omega0$, $\omega0\bar\omega11110$, $\omega\bar\omega\bar\omega\omega1\bar\omega1\omega$, $\bar\omega\omega000\omega\bar\omega1$, $\bar\omega1\omega\bar\omega\omega\omega\bar\omega1$, $1\bar\omega1\bar\omega1\bar\omega0\bar\omega$

D_{41}: 00000111, $00\omega\omega10\bar\omega\bar\omega$, $0\omega0\omega1\bar\omega0\bar\omega$, $\omega00\omega1\bar\omega\bar\omega0$, $\omega\bar\omega\bar\omega0\bar\omega\bar\omega\bar\omega0$, $\bar\omega\omega1\omega\bar\omega\bar\omega1\omega$, $\bar\omega1\omega\omega\bar\omega1\bar\omega\omega$, $1\bar\omega\bar\omega\bar\omega\bar\omega110$

D_{42}: $000\omega\omega\bar\omega\bar\omega1$, $00\omega0\bar\omega\omega1\bar\omega$, $0\omega00\bar\omega1\omega\bar\omega$, $\omega0001\bar\omega\bar\omega\omega$, $\omega\bar\omega\bar\omega1000\omega$, $\bar\omega\omega1\bar\omega00\omega0$, $\bar\omega1\omega\bar\omega0\omega00$, $1\bar\omega\bar\omega\omega\omega000$

D_{43}: $000\omega\omega\bar\omega\bar\omega1$, $00\omega0\bar\omega\omega1\bar\omega$, $0\omega0\omega1\bar\omega0\bar\omega$, $\omega0\omega0\bar\omega1\bar\omega0$, $\omega\bar\omega1\bar\omega00\omega0$, $\bar\omega\omega\bar\omega1000\omega$, $\bar\omega10\bar\omega\omega00\omega$, $1\bar\omega\bar\omega00\omega\omega0$

D_{44}: $000\omega\omega\bar{\omega}\bar{\omega}1$, $00\omega0\bar{\omega}\omega1\bar{\omega}$, $0\omega00\bar{\omega}1\omega\bar{\omega}$, $\omega0001\bar{\omega}\bar{\omega}\omega$, $\omega\bar{\omega}1\bar{\omega}\omega\omega\bar{\omega}1$, $\bar{\omega}\omega\bar{\omega}1\omega\omega1\bar{\omega}$, $\bar{\omega}1\bar{\omega}\omega1\bar{\omega}\omega\omega$, $1\bar{\omega}\omega\bar{\omega}\bar{\omega}1\omega\omega$

D_{45}: $000\omega\omega\bar{\omega}\bar{\omega}1$, $00\omega0\bar{\omega}\omega1\bar{\omega}$, $0\omega0\omega1\bar{\omega}0\bar{\omega}$, $\omega0\omega0\bar{\omega}1\bar{\omega}0$, $\omega\bar{\omega}1\bar{\omega}\omega\omega\bar{\omega}1$, $\bar{\omega}\omega\bar{\omega}1\omega\omega1\bar{\omega}$, $\bar{\omega}10\bar{\omega}\bar{\omega}1\bar{\omega}1$, $1\bar{\omega}\bar{\omega}01\bar{\omega}1\bar{\omega}$

D_{46}: $000\omega\omega\bar{\omega}\bar{\omega}1$, $00\omega0\bar{\omega}\omega1\bar{\omega}$, $0\omega00\bar{\omega}1\omega\bar{\omega}$, $\omega0\bar{\omega}10\omega0\bar{\omega}$, $\omega1\bar{\omega}\bar{\omega}\omega\bar{\omega}\omega1$, $\bar{\omega}\bar{\omega}0\omega1\omega00$, $\bar{\omega}\bar{\omega}\omega\bar{\omega}0\bar{\omega}0\bar{\omega}$, $1\omega1001\bar{\omega}1$

D_{47}: $000\omega\omega\bar{\omega}\bar{\omega}1$, $00\omega0\bar{\omega}\omega1\bar{\omega}$, $0\omega0\omega1\bar{\omega}0\bar{\omega}$, $\omega0\omega0\bar{\omega}1\bar{\omega}0$, $\omega1\bar{\omega}\omega\bar{\omega}\omega\bar{\omega}1$, $\bar{\omega}\bar{\omega}\omega\omega\omega1\bar{\omega}1$, $\bar{\omega}\bar{\omega}\omega\omega1\omega1\bar{\omega}$, $1\omega\omega\bar{\omega}\omega\bar{\omega}1\bar{\omega}$

D_{48}: $000\omega\omega\bar{\omega}\bar{\omega}1$, $00\omega01\bar{\omega}\bar{\omega}\omega$, $0\omega0\bar{\omega}\bar{\omega}0\omega1$, $\omega0\bar{\omega}01\omega0\bar{\omega}$, $\omega1\bar{\omega}10101$, $\bar{\omega}\bar{\omega}0\omega10\omega0$, $\bar{\omega}\bar{\omega}\omega00\omega01$, $1\omega1\bar{\omega}1010$

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